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ON A DOUBLE OF A FREE GROUP

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Dedicated to the memory of my teacher Professor Shimshon A. Amitsur

ABSTRACT

We construct a double of a free group which has an unsolvable occurrence problem, and is (therefore) not LERF.

Let $G = F_1 *_N F_2$ where F_1 and F_2 are free groups of rank 2, and N is a normal subgroup of index 3. Assume further that G is a *double*, i.e. there is an isomorphism ω: $F_1 \to F_2$ which is the identity on N. We exhibit a finitely generated subgroup H of G such that for $g \in G$ there is no algorithmic procedure which decides whether or not $g \in H$. It follows that G is not LERF (*locally extended residually finite = subgroup separable*, i.e. for any element and for any finitely generated subgroup not containing it there is a finite image of the group that separates them).

By contrast, in [GR] some conditions are given that guarantee that the double of a free group over certain finitely generated subgroups is LERF.

Consider the group

$$G = \langle x_1, y_1, x_2, y_2 |\, x_1^3 = x_2^3, y_1^3 = y_2^3, x_1y_1^{-1} = x_2y_2^{-1}, x_1^2y_1^{-1}x_1^{-1} = x_2^2y_2^{-1}x_2^{-1}\rangle.$$

Let $F_1 = \langle x_1, y_1\rangle, F_2 = \langle x_2, y_2\rangle, N = \langle x_1^3, y_1^3, x_1y_1^{-1}, x_1^2y_1^{-1}x_1^{-1}\rangle$. Obviously $G = F_1 *_N F_2$, F_1 and F_2 are free groups of rank 2, $N \triangleleft F_1, N \triangleleft F_2$, and $F_1/N \cong F_2/N \cong C_3$, a cyclic group of order 3.

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Let

$$A = \langle a, b \mid r_1(a,b), \ldots, r_m(a,b)\rangle$$

be any finitely presented 2-generator group. Take

$$s_i = r_i(x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}), \quad 1 \le i \le m;$$
$$H = \langle x_1 y_2^{-1}, x_1^2 y_2^{-1} x_1^{-1}, x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle.$$

Lemma 1: *$H/H \cap N$ is a free group of rank 2 generated by the images of $x_1 y_2^{-1}$ and $x_1^2 y_2^{-1} x_1^{-1}$.*

Proof: Identify $H/H \cap N$ with a subgroup of $G/N \cong C_3 * C_3 = \langle c_1, c_2 \mid c_1^3, c_2^3\rangle$, where the isomorphism takes x_1 and y_1 to c_1 and x_2 and y_2 to c_2. The elements $c_1 c_2^{-1}$ and $c_1^2 c_2^{-1} c_1^{-1}$ generate a free subgroup of rank 2 of $C_3 * C_3$, whence the lemma.

Define $\alpha: G \to G$ by $\alpha(x_1) = \alpha(x_2) = x_1, \alpha(y_1) = \alpha(y_2) = y_2$. Consider the inner automorphisms of G induced by x_1 and x_2. Obviously their restrictions to N coincide, and the same is true for y_1 and y_2. Therefore

Lemma 2: *For $g \in G, h \in N$, we have $ghg^{-1} = \alpha(g)h\alpha(g)^{-1}$.*

Lemma 3: *$H \cap N = \langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle^N$.*

Proof: Clearly, $\langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle \subset H \cap N$, so $\langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle^N \subset N$. Let $h \in \langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle$ and $g \in N$. We have

$$\alpha(N) = \alpha(\langle x_1^3, y_1^3, x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}\rangle) = \langle x_1^3, y_2^3, x_1 y_2^{-1}, x_1^2 y_2^{-1} x_1^{-1}\rangle \subset H,$$

so by Lemma 2,

$$ghg^{-1} = \alpha(g)h\alpha(g)^{-1} \in H,$$

therefore $\langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle^N \subset H \cap N$.

For the proof of the reverse inclusion put $L = \langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m\rangle^N$, and $z_1 = x_1 y_2^{-1}, z_2 = x_1^2 y_2^{-1} x_1^{-1}$. Every element $h \in H$ can be written in the form

$$(*) \qquad h = t_0 z_{i_1}^{\epsilon_1} t_1 z_{i_2}^{\epsilon_2} t_2 \cdots t_{n-1} z_{i_n}^{\epsilon_n} t_n$$

where $t_j \in L, i_j = 1$ or 2, $\epsilon_j = 1$ or -1. It follows from Lemma 1 that $h \in H \cap N$ if and only if

$$z_{i_1}^{\epsilon_1} z_{i_2}^{\epsilon_2} \cdots z_{i_n}^{\epsilon_n} = 1.$$

Given $h \in H \cap N$, assume that in $(*)$ n is minimal possible. We claim that $n = 0$. Otherwise, there is a subword of $(*)$ of the form $z_i^{\epsilon} t z_i^{-\epsilon}$, $t \in L$. Then for $u_1 = x_1 y_1^{-1}, u_2 = x_1^2 y_1^{-1} x_1^{-1}$, by Lemma 2,

$$z_i^{\epsilon} t z_i^{-\epsilon} = \alpha(u_i^{\epsilon}) t \alpha(u_i^{\epsilon})^{-1} = u_i^{\epsilon} t u_i^{-\epsilon} = t' \in L,$$

because $u_1, u_2 \in N$. Substituting t' for $z_i^{\epsilon} t z_i^{-\epsilon}$ we decrease n, which is a contradiction. Therefore $n = 0, h = t_0 \in L$, so $H \cap N \subset L = \langle x_1^3, y_2^3, s_1, s_2, \ldots, s_m \rangle^N$.

LEMMA 4: *$N/H \cap N \cong A$.*

Proof: Note that $x_1^3, y_2^3, x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}$ freely generate N. By Lemma 3,

$$\begin{aligned}
N/H \cap N =& \langle x_1^3, y_2^3, x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1} \mid x_1^3, y_2^3, s_1, \ldots, s_m \rangle \\
\cong& \langle x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1} \mid s_1, \ldots, s_m \rangle \\
=& \langle x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1} \mid r_1(x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}), \ldots, r_m(x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}) \rangle \\
\cong& \langle a, b \mid r_1(a,b), \ldots, r_m(a,b) \rangle = A.
\end{aligned}$$

We conclude that for any word w on the letters a, b, the equality $w(a,b) = 1$ holds in A if and only if $w(x_1 y_1^{-1}, x_1^2 y_1^{-1} x_1^{-1}) \in H \cap N$. Taking A with an unsolvable word problem (see, e.g., [R], Ch. 12) we obtain that for $g \in G$ there is no algorithmic procedure which decides whether or not $g \in H$. Thus we have proved the following

THEOREM: *G has an unsolvable occurrence problem.*

Note that it was shown by K. A. Mikhailova [M] that the free product of groups with a solvable occurrence problem also has a solvable occurrence problem.

COROLLARY: *G is not LERF.*

Indeed, it is well-known that for every LERF group the occurrence problem is solvable.

References

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